Observer Design

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1 Introduction

This note is about observer, mainly focused on the high-gian observer which is suitable for most of the simple dynamic systems we met in the real world.

2 Triangular form

the general form can be written as:

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \Phi_1(u, \xi_1) \\ ... \\ \dot{\xi}_i = \xi_i + \Phi_i(u, \xi_1, \xi_2, ..., \xi_i) \\ ... \\ \dot{\xi}_m = \Phi_m(u, \xi) \\ y = \xi_1 \end{cases}$$

where for all 1,2,...m

$$\xi_i \in R^d$$

$$\xi = (\xi_1, \xi_2, .., \xi_m)$$

Also, the function Φ_i need to be Lipschitz, that is to say

$$|\Phi_i(u, \xi_{1a}, \xi_{2a}, ..., \xi_{ia}) - \Phi_i(u, \xi_{1b}, \xi_{2b}, ..., \xi_{ib})| \leq a \sum_{j=1}^{i} |\xi_{ja} - \xi_{jb}|$$

once we get the general form, we can get the observer:

$$\begin{cases} \dot{\hat{\xi}}_1 = \hat{\xi}_2 + \Phi_1(u, \hat{\xi}_1) - Lk_1(\hat{\xi}_1 - y) \\ ... \\ \dot{\hat{\xi}}_i = \hat{\xi}_i + \Phi_i(u, \hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_i) - L^i k_i(\hat{\xi}_1 - y) \\ ... \\ \dot{\hat{\xi}}_m = \Phi_m(u, \hat{\xi}) - L^m k_m(\hat{\xi}_1 - y) \end{cases}$$

now let's turn to the parameter: $\Phi$is defined as before, $\hat{\xi}$ means the state in the observer, that is , the ' simulated ' data.
L is the high-gain parameter, it measures the 'force' of the boundary when $\hat{\xi}_1$ deviate from y. (same as K in linear form)
$(k_1, ...k_m) \in R^n$satisfied the roots of:

$$s^m + k_m s^{m-1} + ... + k_2 s + k_1$$

has strictly negative real part.also we need Φ to be lipschitz.
then we can satisfied that it follows

$$|\hat{\xi}(t) - \xi(t)| \leq \beta |\hat{\xi}(t_0) - \xi(t_0)| e^{-(2\lambda L - a\sqrt{\frac{m}{L^2-1}}L\sqrt{\frac{\alpha_2}{\alpha_1}})}$$

actually, this boundary is not the best form. If we define ϵ as

$$\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_m), \quad e_i = L^{i-1}\epsilon_i$$

then it can be write in this form:

$$|\hat{\epsilon}(t) - \epsilon(t)| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} |\hat{\epsilon}(t_0) - \epsilon(t_0)| e^{-(2\lambda L - a\sqrt{\frac{m}{L^2-1}}L\sqrt{\frac{\alpha_2}{\alpha_1}})}$$

where a is the lipschitz constant,$\lambda, \alpha_1, \alpha_2$can be found in the proof.
proof
We consider a more general form:

$$\begin{cases} \dot{\hat{\xi}}_1 = \hat{\xi}_2 + \tilde{\Phi}_1(u, \hat{\xi}_1) - Lk_1(\hat{\xi}_1 - y) \\ ... \\ \dot{\hat{\xi}}_i = \hat{\xi}_i + \tilde{\Phi}_i(u, \hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_i) - L^i k_i(\hat{\xi}_1 - y) \\ ... \\ \dot{\hat{\xi}}_m = \tilde{\Phi}_m(u, \hat{\xi}) - L^m k_m(\hat{\xi}_1 - y) \end{cases}$$

while Φ indicate the real dynamics and $\tilde{\Phi}$ indicate the approximation dynamics. They satisfied

$$|\Phi(x) - \tilde{\Phi}(x)| < M$$

We define matrix A as

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and also the matrix $\mathcal{L}$:

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & L^2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & L^{m-1} \end{bmatrix}$$

$$C = \begin{bmatrix} I_{d_y} & 0 & \cdots & 0 \end{bmatrix}$$
$$K = \begin{bmatrix} k_1 I \\ k_2 I \\ \cdots \\ k_m I \end{bmatrix}$$

Then we can rewrite the dynamics into

$$\dot{\hat{\xi}} = A\hat{\xi} + \tilde{\Phi}(\hat{\xi}) - L\mathcal{L}KCe$$
$$e = \hat{\xi} - \xi$$

And the original dynamics is

$$\dot{\xi} = A\xi + \Phi(\xi)$$

So make the difference we have:

$$\dot{e} = Ae + \tilde{\Phi}(\hat{\xi}) - \Phi(\xi) - L\mathcal{L}KCe = Ae + \Delta\Phi - L\mathcal{L}KCe$$

$$\begin{aligned}\mathcal{L}^{-1}\dot{e} &= \mathcal{L}^{-1}A\mathcal{L}(\mathcal{L}^{-1}e) + \mathcal{L}^{-1}\Delta\Phi - LKC\mathcal{L}(\mathcal{L}^{-1}e) \\ &= L\mathcal{L}^{-1}\mathcal{L}A(\mathcal{L}^{-1}e) + \mathcal{L}^{-1}\Delta\Phi - LKC(\mathcal{L}^{-1}e)\end{aligned}$$

define ϵ as $\mathcal{L}^{-1}e$

$$\frac{1}{L}\epsilon = (A - KC)\epsilon + \frac{1}{L}\mathcal{L}^{-1}\Delta\Phi$$

We can choose K so that A-KC is Hurwitz, so

$$\exists P, \quad (A - KC)^T P + P(A - KC) \leq -4\lambda P$$

define $V = \epsilon^T P \epsilon$ Then

$$\begin{aligned}\frac{1}{L}\dot{V}(\epsilon) &= 2\epsilon^T P((A - KC)\epsilon + \frac{1}{L}\mathcal{L}^{-1}\Delta\Phi) \\ &\leq -4\lambda V(\epsilon) + 2\epsilon^T P\frac{1}{L}\mathcal{L}^{-1}\Delta\Phi \\ &= -4\lambda V(\epsilon) + 2\epsilon^T P\frac{1}{L}\mathcal{L}^{-1}(\tilde{\Phi}(\hat{\xi}) - \Phi(\hat{\xi}) + \Phi(\hat{\xi}) - \Phi(\xi)) \\ &\leq -4\lambda V(\epsilon) + 2||\epsilon^T P||\frac{1}{L}\mathcal{L}^{-1}M + 2\epsilon^T P\frac{1}{L}\mathcal{L}^{-1}(\Phi(\hat{\xi}) - \Phi(\xi))\end{aligned}$$

$$|\Phi_i(\hat{\xi}) - \Phi_i(\xi)| \leq a\sum_{j=1}^{i} ||\hat{\xi}_j - \xi_j||$$

here we use the sum of L_2 norm because for linear system the sum of L_2 norm is tight for lipschitz condition.

Then

$$\begin{aligned}
||\mathcal{L}^{-1}(\Phi(\hat{\xi}) - \Phi(\xi))||^2 &= \sum_{i=1}^{m}(\sum_{j=1}^{i} \frac{a}{L^{i-1}}||\hat{\xi}_j - \xi_j||)^2 \\
&\leq \sum_{i=1}^{m} \frac{a^2}{L^{2i-2}} m(\sum_{j=1}^{i} ||\hat{\xi}_j - \xi_j||^2) \\
&= a^2 m \sum_{i=1}^{m}(\sum_{j=1}^{i} \frac{1}{L^{2i-2}}||\hat{\xi}_j - \xi_j||^2) \\
&= a^2 m \sum_{j=1}^{m}(\sum_{i=j}^{m} \frac{1}{L^{2i-2}})||\hat{\xi}_j - \xi_j||^2 \\
&\leq a^2 m \sum_{j=1}^{m} \frac{L^2}{L^2-1}\frac{1}{L^{2j-2}}||\hat{\xi}_j - \xi_j||^2 \\
&= a^2 m \sum_{j=1}^{m} \frac{L^2}{L^2-1}||\epsilon_i||^2 \\
&= a^2 m \frac{L^2}{L^2-1}||\epsilon||^2
\end{aligned}$$

So finally we have

$$\dot{V}(\epsilon) \leq -4\lambda L V(\epsilon) + 2||\epsilon^T P||M + 2a\sqrt{m\frac{L^2}{L^2-1}}||\epsilon^T P||||\epsilon||$$

And we have

$$\begin{aligned}
||\epsilon^T P|| &\leq \sqrt{\epsilon^T P \epsilon}\sqrt{\alpha_2} \\
||\epsilon|| &\leq \frac{1}{\sqrt{\alpha_1}}\sqrt{\epsilon^T P \epsilon}
\end{aligned}$$

α_1,α_2 is the min and max eigen value of P.
So we plug in,

$$\begin{aligned}
\dot{V}(\epsilon) &\leq -(4\lambda L - 2a\sqrt{m\frac{L^2}{L^2-1}}\sqrt{\frac{\alpha_2}{\alpha_1}})V(\epsilon) + 2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{V(\epsilon)M}{||\epsilon||} \\
&= -2rV(\epsilon) + 2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{M}{||\epsilon||}V(\epsilon)
\end{aligned}$$

if $2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{M}{||\epsilon||} < r$

$$\dot{V}(\epsilon) \leq -rV(\epsilon)$$

it is easy to show error decrease expenentially. otherwise if $2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{M}{||\epsilon||} \geq r$,we have $||\epsilon|| \leq 2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{M}{r}$ Finally it leads to:

$$|\epsilon(t)| \leq \max\{\sqrt{\frac{\alpha_2}{\alpha_1}}|\epsilon(t_0)|e^{-(\lambda L - \frac{aL}{2}\sqrt{\frac{m}{L^2-1}}\sqrt{\frac{\alpha_2}{\alpha_1}})}, 2\sqrt{\frac{\alpha_2}{\alpha_1}}\frac{M}{-(2\lambda L - a\sqrt{\frac{m}{L^2-1}}L\sqrt{\frac{\alpha_2}{\alpha_1}})}\}$$

As for the result we get:

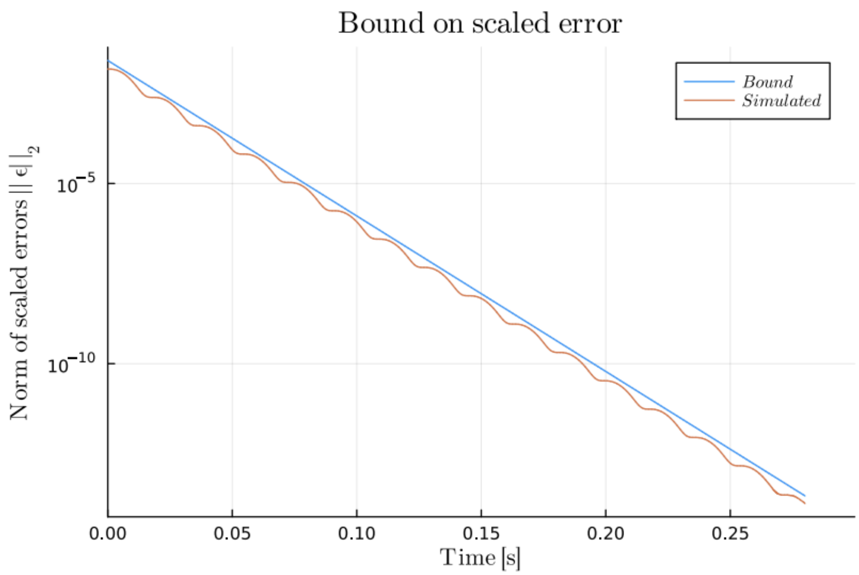

Figure 1: error bound

2.1 cart-pole

first let us look at the dynamic of cart-pole system:

$$\ddot{x} = \frac{1}{m_c + m_p sin^2\theta}(u + m_p sin\theta(l\dot{\theta}^2 + gcos\theta))$$

$$\ddot{\theta} = \frac{1}{l(m_c + m_p sin^2\theta)}[-ucos\theta - m_p l\dot{\theta}^2 cos\theta sin\theta - (m_c + m_p)gsin\theta]$$

we can rewrite the dynamics into triangular form:

$$\begin{cases} \dot{\xi_1} = \xi_2 \\ \dot{\xi_2} = \begin{bmatrix} \frac{u + m_p sin\theta gcos\theta}{m_c + m_p sin^2\theta} \\ \frac{-ucos\theta - (m_c + m_p)gsin\theta}{l(m_c + m_p sin^2\theta)} \end{bmatrix} + \begin{bmatrix} \frac{m_p sin\theta l\dot{\theta}^2}{m_c + m_p sin^2\theta} \\ \frac{-m_p l\dot{\theta}^2 sin\theta cos\theta}{l(m_c + m_p sin^2\theta)} \end{bmatrix} = \Phi(u, \xi_1, \xi_2) \end{cases}$$

where

$$\xi_1 = \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$$

so we get the observer as:

$$\begin{cases} \dot{\hat{\xi}}_1 = \hat{\xi}_2 - Lk_1(\hat{\xi}_1 - y) \\ \dot{\hat{\xi}}_2 = \Phi(u, \hat{\xi}_1, \hat{\xi}_2) - L^2 k_2(\hat{\xi}_1 - y) \end{cases}$$

as we discussed above, k_1, k_2need to satisfy some rules. in this problem they only need to be positive.
unfortunately, this function is **not** global lipschitz, unless we can bounded the $\dot{\theta}$

proof
we can define

$$\Phi_1 = \begin{bmatrix} \frac{u+m_p sin\theta g cos\theta}{m_c+m_p sin^2\theta} \\ \frac{-ucos\theta-(m_c+m_p)gsin\theta}{l(m_c+m_p sin^2\theta)} \end{bmatrix}, \Phi_2 = \begin{bmatrix} \frac{m_p sin\theta l\dot{\theta}^2}{m_c+m_p sin^2\theta} \\ \frac{-m_p\dot{\theta}^2 sin\theta cos\theta}{m_c+m_p sin^2\theta} \end{bmatrix}$$

then there must be

$$\Phi_1 + \Phi_2 = \Phi$$

note that when we calculate $|\hat{\xi} - \xi|$when we proof the convegence of the system, Φ_1 is only the function of θ, so it can disappear when we calculate the difference. So the only task we need to du is to proof Φ_2 is lipschitz.

$$\begin{aligned} |\Phi_2(u,\xi_a) - \Phi_2(u,\xi_b)| &= \frac{m_p}{m_c + m_p sin^2\theta}\sqrt{l^2(sin\theta_a\dot{\theta}_a^2 - sin\theta_b\dot{\theta}_b^2)^2 + (cos\theta_a sin\theta_a\dot{\theta}_a^2 - cos\theta_b sin\theta_b\dot{\theta}_b^2)^2} \\ &\leq \frac{m_p max(l,1)}{m_c + m_p}\sqrt{(sin\theta_a\dot{\theta}_a^2 - sin\theta_b\dot{\theta}_b^2)^2 + (cos\theta_a sin\theta_a\dot{\theta}_a^2 - cos\theta_b sin\theta_b\dot{\theta}_b^2)^2} \end{aligned}$$

if we have $|\dot{\theta}| < K$

$$\begin{aligned} &(sin\theta_a\dot{\theta}_a^2 - sin\theta_b\dot{\theta}_b^2)^2 \\ =&((sin\theta_b + (sin\theta_a - sin\theta_b))(\dot{\theta}_b^2 + (\dot{\theta}_a^2 - \dot{\theta}_b^2)) - sin\theta_b\dot{\theta}_b^2)^2 \\ =&((sin\theta_a - sin\theta_b)(\dot{\theta}_a^2 - \dot{\theta}_b^2) + sin\theta_b(\dot{\theta}_a^2 - \dot{\theta}_b^2) + \dot{\theta}_b^2(sin\theta_a - sin\theta_b))^2 \\ \leq&(3(\dot{\theta}_a^2 - \dot{\theta}_b^2) + \dot{\theta}_b^2(sin\theta_a - sin\theta_b))^2 \\ \leq&(6K|\dot{\theta}_a - \dot{\theta}_b| + K^2|\theta_a - \theta_b|)^2 \\ \leq&max(6K, K^2)^2(|\dot{\theta}_a - \dot{\theta}_b| + |\theta_a - \theta_b|)^2 \end{aligned}$$

also as the same, $cos\theta_b sin\theta_b \dot{\theta}_b^2 = \frac{1}{2} sin(2\theta)\dot{\theta}_b^2$

$$
\begin{aligned}
&(cos\theta_a sin\theta_a \dot{\theta}_a^2 - cos\theta_b sin\theta_b \dot{\theta}_b^2)^2 \\
=&\frac{1}{4}((sin2\theta_b + (sin2\theta_a - sin2\theta_b))(\dot{\theta}_b^2 + (\dot{\theta}_a^2 - \dot{\theta}_b^2)) - sin2\theta_b \dot{\theta}_b^2)^2 \\
=&((sin2\theta_a - sin2\theta_b)(\dot{\theta}_a^2 - \dot{\theta}_b^2) + sin2\theta_b(\dot{\theta}_a^2 - \dot{\theta}_b^2) + \dot{\theta}_b^2(sin2\theta_a - sin2\theta_b))^2 \\
\leq&(3(\dot{\theta}_a^2 - \dot{\theta}_b^2) + \dot{\theta}_b^2(sin2\theta_a - sin2\theta_b))^2 \\
\leq&(6K|\dot{\theta}_a - \dot{\theta}_b| + 2K^2|\theta_a - \theta_b|)^2 \\
\leq&max(6K, 2K^2)^2(|\dot{\theta}_a - \dot{\theta}_b| + |\theta_a - \theta_b|)^2
\end{aligned}
$$

let $r^2 = max(max(6K, K^2)^2, max(6K, 2K^2)^2)$,then

$$
\begin{aligned}
|\Phi_2(u,\xi_a) - \Phi_2(u,\xi_b)| &= \frac{m_p}{m_c + m_p sin^2\theta}\sqrt{l^2(sin\theta_a\dot{\theta}_a^2 - sin\theta_b\dot{\theta}_b^2)^2 + (cos\theta_a sin\theta_a\dot{\theta}_a^2 - cos\theta_b sin\theta_b\dot{\theta}_b^2)^2} \\
&\leq \frac{m_p max(l,1)r}{m_c + m_p}(|\dot{\theta}_a - \dot{\theta}_b| + |\theta_a - \theta_b|) \\
&\leq \frac{m_p max(l,1)r}{m_c + m_p}(|\xi_{1a} - \xi_{1b}| + |\xi_{2a} - \xi_{2b}|)
\end{aligned}
$$

so that is to say it is lipschitz.
actually,thinks of$|\hat{\theta_a}^2 - \hat{\theta_b}^2|$. If we fixed $|\theta_a - \theta_b|$,then if we push $\theta_a \longrightarrow \infty$,then $|\hat{\theta_a}^2 - \hat{\theta_b}^2|$ will be ∞.

here are some result:

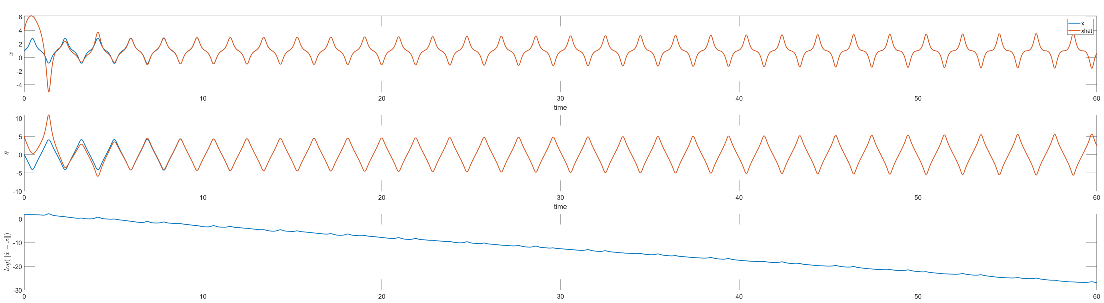

Figure 2: cart-pole

2.2 quadrotor

Original dynamics: Assume there are body frame with the quadrotor and the world frame not change. The rotation matrix from world frame to body frame is R. Then dynamics should be:

$$\begin{cases} \dot{x_c} = v_c \\ \dot{v_c} = M^{-1}(Mg + Rf_b) \\ \dot{q} = q \odot \tilde{w} \\ \dot{\omega} = J^{-1}(\tau - \omega \times J\omega) \end{cases}$$

but it dont satisfied high-gain form, we must turn it to another form

$$\begin{cases} \dot{x_c} = v_c \\ \dot{v_c} = M^{-1}(Mg + Rf_b) \\ \dot{q} = v_q \\ \dot{v_q} = q \odot \begin{bmatrix} q^T q \\ \frac{1}{2}\dot{q} \end{bmatrix} \end{cases}$$

and the result:

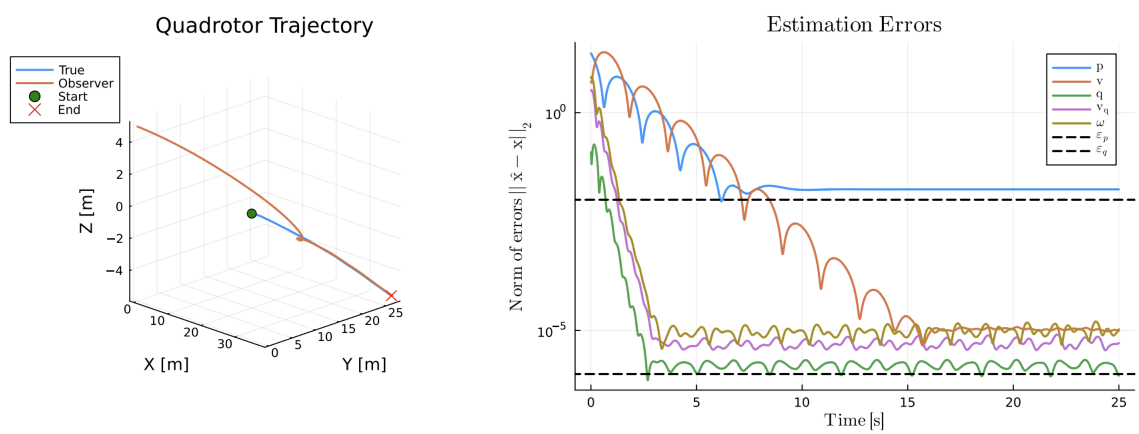

Figure 3: quadrator

2.3 noise estimation

If there is some noise for estimation, that is

$$y = \xi_1 + w, \quad ||w|| < \epsilon_e$$

then the convegence of high-gain changed: if we define

$$r = 2\lambda L - a\sqrt{\frac{m}{L^2 - 1}} L \sqrt{\frac{\alpha_2}{\alpha_1}}$$

then

$$|\hat{\epsilon}(t)-\epsilon(t)| \leq max\{\sqrt{\frac{\alpha_2}{\alpha_1}}|\hat{\epsilon}(t_0)-\epsilon(t_0)|e^{-\frac{1}{2}(t-t_0)}, \frac{2||PK||\epsilon_e L}{\alpha_1 r}\}$$

3 region of attraction(ROA)

from the paper, theory 4.4 guarantee that if we have the dynamics and the auxiliary function satisfied some assumption(from H0 to H6), then we can solve the lypunov function from a PDE and determined the region of attraction by define

$$\Omega = \{x|V(x) < +\infty\}$$

so the first step is to find an auxiliary function for the dynamics.

3.1 some conditions in the theory

we go to the H0-H2, which is some properties for the dynamics it self.
assump the dynamics is of this form:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \Phi_1(\xi_1,\xi_2) + \Phi_2(\xi_1,\xi_2)u \end{cases}$$

assume Φ_1, Φ_2 to be local lipschitz, Φ_2 is bounded and not to be 0 at (0,0), and for every R ,$\Phi_1(\xi_1,\xi_2) \leq C_d||\xi||^{2p}, p \in \mathbf{Z} \quad for ||\xi|| \geq d$
the form can be written as

$$\dot{x} = f(x,u)$$

(all of the four example satisfied this form)
assume

$$u \in U = \{u|||u|| < 1\}$$

(H0)

$$||f(x,u) - f(y,u)|| \leq C_R(1 + ||u||)||x - y||$$

$$R = max\{||x||, ||y||\}$$

(just as what we get from the high-gain form, only need Φ_1 and Φ_2 to be lipschitz, actually, because the C is depend on R, the local lipschitz is ok)
(H1)

$$f(0,0) = 0$$

need some transform to make (0,0), the origin, to be a equilibrium point.
(H2')
there exist a open ball B(0,r) and c, σ, s.t. for every $x \in B(0,r)$ there exist bounded u(that is to say, u $\in L^\infty([0,+\infty], U)$ and

$$||\phi(t,x,u)|| + ||u(t)|| \leq ce^{\sigma t}||x||$$

*Note:
ϕ is the solution of initial state x and control input u(t),so the equation is to say , we "can" get a u that make the system exponential decrease.

proof: only a high-level idea, note that

$$f(0,0) = 0$$

so around the equilibrium point, Φ_1 can be small, thus we can choose a sufficient small u that make the system like

$$\dot{x} = Ax$$

and A is hurwitz. this need Φ_2 not to be 0 around the origin, what is satisfied for pendulum,cartpole and arcbots
now we get the ϕ is expenential decreasing, but what about u? actually when we made a taylor expansion of Φ_1, so if the Φ_1 is continous differentiable, then we can get

$$||\Phi_1|| \leq C||x||$$

near the neigor of origin, then somehow(with some scaling) we can get

$$||u|| \leq D||x||$$

then the (H2') satisfied.
after that we go to the property H3-H6, which is related to the auxiliary function g.
(H3)
first, g has to satisfied (H0), but this is easy, g to be local lipschitz is ok.
second, g will not go to ∞ in a sphere and always positive
these condition are not difficult to satisfy.
*Note
actually the auxiliary function g is $-\dot{V}$, which V is the control lypunov function. so positive is essential. more details will show later.
(H4')

$$g(x,u) \leq C(||x|| + ||u||)^{\mu}$$

if g is a polymial, this is also easy to satisfied.
(H6)

$$\hat{g} = \frac{g(x,u)}{1 + ||f(x,u)||}$$

$\hat{g} \to +\infty$ as $||u|| \to +\infty$ for every x
because we assume f is a linear function of u, so we add a quadratical term of u on g is ok.
finally it comes (H5)

(H5)
$\exists \quad C, \quad s.t.$

$$g(x,u) \geq ||\Phi_1(\xi_1,\xi_2)|| + C||u||$$

when $||x|| \geq 2r, ||u|| \geq 2cr,\ c, r$ is the parameter in H2'
since we add a quadratical term of u on g in H5,the part of $C||u||$ is easy to satisfied, and with the condition

$$||\Phi_1(\xi_1,\xi_2)|| \leq C_d||\xi||^{2p}, p \in \mathbf{Z} \quad for||\xi|| \geq d$$

we can choose $g(x,u) = (c_d + 1)||x||^{2p} + b||u||^2$

3.2 how to solve the equation

according to the theory 4.4, we can get:
we can solve equation

$$\sup_{u \in U}\{-DV(x)f(x,u) - g(x,u)\} = 0$$

to get the unique viscosity solution V(x) as a lyapunov function
*Note
the equation inside is

$$-\dot{V}(x) - g(x,u)$$

, so this equation is to say we can find a control input u that make

$$\dot{V}(x) = -g(x,u)$$

, so that we can get V is indeed a lyapunov function from g's properties.
if we do not need to know how to solve the exactly form of V, then the region of attraction under control input is defined by

$$\Omega = \{x|V(x) < +\infty\}$$

so

$$\Omega = \{x|\dot{V}(x) < +\infty\}$$

(this is a relaxation)
when will $\dot{V}$ go to ∞? that is, when f(x,u) = 0 and g(x,u)$\neq$0, when we plug in $g(x,u) = (c_d + 1)||x||^{2p} + b||u||^2$ and apply in the pendulum, the solution will be two points: you stretch the pendulum, i.e. the control input is 1 or -1 and the system is equilibrium.but this is not in Ω
*Note
i think the reason why we solve some other points is, the solution is a viscosity one, so sometime it do not have a gradient, and that is the two situation above. without these two there is no points that $\dot{V}$ go to ∞. it makes sense that everywhere we can apply a u = -$\frac{\dot{\theta}}{||\dot{\theta}||}$ that make the system finally converge.